$$\sin(A+B) = \sqrt{(\sin A \cos B + \cos A \sin B)^2}.$$

$$= \sqrt{\sin^2 A(1-\sin^2 B) + (1-\sin^2 A)\sin^2 B + 2\sin A\cos A\sin B\cos B}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B(\cos A\cos B - \sin A\sin B)}$$

$$= \sqrt{\sin^2 A + \sin^2 B + 2\sin A\sin B\cos(A+B)}.$$

From the given equations we have

$$\sin(\theta+\alpha) = \frac{m}{r} \qquad \text{and} \qquad \sin\left[\frac{\pi}{2} - (\theta+\beta)\right] = \frac{n}{r}.$$

Hence letting

$$A = (\theta+\alpha) \qquad \text{and} \qquad B = \left[\frac{\pi}{2} - (\theta+\beta)\right],$$

we have

$$(A+B) = \left[\frac{\pi}{2} + (\alpha-\beta)\right].$$

Substituting, we have

$$\sin\left[\frac{\pi}{2} + (\alpha-\beta)\right]$$

$$= \sqrt{\sin^2(\theta+\alpha) + \sin^2\left[\frac{\pi}{2} - (\theta+\beta)\right] + 2\sin(\theta+\alpha)\sin\left[\frac{\pi}{2} - (\theta+\beta)\right]\cos\left[\frac{\pi}{2} + (\alpha-\beta)\right]},$$

or

$$\cos(\alpha-\beta) = \sqrt{\frac{m^2}{r^2} + \frac{n^2}{r^2} - 2\frac{m}{r}\frac{n}{r}\sin(\alpha-\beta)}.$$

Solving for $r$, we have

$$r = \frac{\sqrt{m^2+n^2-2mn\sin(\alpha-\beta)}}{\cos(\alpha-\beta)}.$$

Also solved by B. J. Brown, Herbert N. Carlton, Nathan Altshiller, C. E. Horne, C. N. Schmall, Paul Capron, J. A. Caparo, Frank Irwin, Frank R. Morris, S. A. Joffe, V. M. Spunar, L. G. Weld, George W. Hartwell, Elijah Swift, A. W. Smith, Joseph B. Reynolds, Richard Morris, Walter C. Eells, Albert N. Nauer, Elmer Schuyler, and A. M. Harding.

**428. Proposed by FRANK IRWIN, University of California.**

If the roots of the equation

$$x^n - na_1x^{n-1} + \binom{n}{2}a_2x^{n-2} + \cdots = 0$$

are all real, the condition that they should all be equal is $a_1^2 = a_2$. A proof of the sufficiency of the condition is readily obtained from a consideration of derivatives. A proof is desired not based on such considerations.

### Solution by Elijah Swift, University of Vermont.

We are to show that if the roots are all real, and if $a_1^2 = a_2$, the roots are all equal. For that purpose write the above equation in the form

$$(x - a_1)^n + \binom{n}{2}(a_2 - a_1^2)x^{n-2} + \varphi(x) = 0,$$

where $\varphi(x)$ is a polynomial of degree $n - 3$ at most. Suppose $a_2 = a_1^2$, and write $y = x - a_1$. The equation becomes

$$y^n + \varphi(y + a_1) \equiv y^n + c_3y^{n-3} + \cdots + c_n = 0.$$

If all the $c$'s are zero, the theorem is true. If not, let $c_k$ be the first coefficient not zero, and $c_e$ the last. The equation is then

$$y^n + c_ky^{n-k} + \cdots + c_ey^{n-l} = 0.$$

Apply Descartes's rule to this equation. If we call the left hand side $f(y)$, $f(y)$ and $f(-y)$ can have together at most $n - k + 1 - l$ variations of sign ($k$ odd), or $n - k + 2 - l$ ($k$ even), hence at most $n - 2 - l$. The equation has, therefore, at least $2 + l$ zero or complex roots. Exactly $l$ roots are zero, however, hence it must have 2 complex. This contradicts our hypothesis that the roots were all real. Hence all the $c$'s are zero and $f(y) \equiv y^n$, and the equation in $x$ is $(x - a_1)^n = 0$.

Also solved by LAENAS G. WELD.

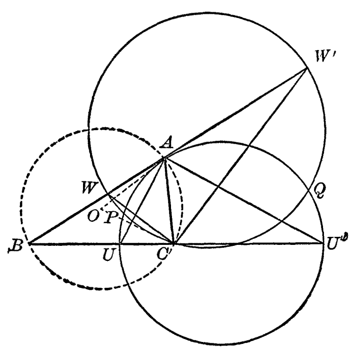
## GEOMETRY.

**456. Proposed by J. W. CLAWSON, Ursinus College.**

The interior and exterior bisectors of the angles $A$, $B$, $C$ of a triangle meet the opposite sides in $U$, $U'$; $V$, $V'$; $W$, $W'$ respectively. Circles are drawn on $UU'$, $VV'$, $WW'$ as diameters (Circles of Apollonius). Prove that (1) these three circles have a common chord. (2) The center of the circumcircle lies on this common chord.

### SOLUTION BY PROPOSER.

1. $A(BC, UU')$ is a harmonic pencil; so $(BC, UU')$ is a harmonic range. If $T$ be any point on the circle having $UU'$ as diameter, $T(BC, UU')$ is a harmonic




pencil. But $UTU'$ is a right angle. Therefore $TU$ bisects $\angle BTC$. Hence,

$$BT : TC = BU : UC = BA : AC.$$